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Recursive Methods for Evaluating Expected Penalties at Claim Instants

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Abstract This paper considers the finite-time ruin problem for the classical risk model where the ruin occurs at claim instants. In order to study more ruin related quantities at claim (ruin) instants, the joint trivariate probability density function of the surplus after the second last claim before ruin, the surplus prior to ruin and the deficit immediately after ruin at these ruin moments is derived. The corresponding expected penalty on three variates as a function of the initial surplus is introduced and the recursive method for evaluating the expected penalty function at claim instants is proposed. Finally, numerical illustrations are given when the claim amounts are exponentially distributed.

Keywords Classical risk model · Finite-time ruin probability · Surplus before ruin · Deficit after ruin · Expected penalty function

Mathematics Subject Classification (2000) 62P05 · 60G40 · 91B30

1 Introduction

The ultimate ruin (solvency) probability has been an active research area of insurance mathematics from the early days of Lundberg. It has simpler analytical expressions for models of common interests both in theory and practice. The so-called finite-time ruin probability is the probability that ruin occurs within a finite time horizon. Unfortunately, it is difficult to express the finite-time ruin probabilities in a closed form, even in the classical compound Poisson model (Rolski et al. 1999). Hence, the approximation of the finite-time ruin probability and related quantities becomes essential and important in practice for risk management.

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Consider a classical risk model in which the surplus process $\{U(t); t \geq 0\}$ with initial surplus $u \geq 0$ is given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i \,, \qquad t \geq 0 \,, \tag{1.1}$$

where c is the constant premium rate and $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random variables representing the individual claim amounts, with common probability distribution function (d.f.) F, density f, mean $1/\mu$ and Laplace transform (LT) $\hat{f}$. The counting process $\{N(t);\, t \geq 0\}$ denotes the number of claims up to time t and is defined as $N(t) = \max\{k : Z_1 + Z_2 + \cdots + Z_k \leq t\}$, where the interclaim times Z_i's are assumed to be independent and exponentially distributed random variables with mean $1/\lambda_1$. That is, $\{N(t);\, t \geq 0\}$ is a Poisson process with parameter $\lambda_1 > 0$. We further assume that $\{Z_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ are independent.

Let $\tau = \inf\{t \mid U(t) < 0\}$ be the time of ruin with $\tau = \infty$ if ruin does not occur. Define the probability of ruin by time t by

$$\psi(u,t) = \Pr\{\tau \leq t \mid U(0) = u\} \,, \qquad u, t \geq 0 \,,$$

and the probability of non-ruin up to time t by $\sigma(u,t) = 1 - \psi(u,t)$. Then the probability of ultimate ruin as a function of the initial surplus $U(0) = u \geq 0$ is defined by

$$\psi(u) = \psi(u,\infty) = \Pr\{\tau < \infty \mid U(0) = u\} \,, \qquad u \geq 0 \,.$$

We assume that the positive loading condition holds, i.e.,

$$c > \frac{\lambda_1}{\mu} \,, \tag{1.2}$$

and hence $\psi(u) < 1$. It is well-known (Gerber 1979) that the non-ruin function $\sigma(u,t)$ satisfies the following partial integro-differential equation

$$c \, \frac{\partial \sigma(u,t)}{\partial u} = \frac{\partial \sigma(u,t)}{\partial t} + \lambda_1 \, \sigma(u,t) - \lambda_1 \int_0^u \sigma(u-y,t) dF(y) \,, \qquad u, t \geq 0 \,. \tag{1.3}$$

Solving equation (1.3) in general is difficult though is not impossible. Some techniques are employed and calculating algorithms are proposed to obtain and to approximate the finite-time ruin probabilities. Among related research in the literature of ruin theory, An algorithm for approximating by discretizing and recursion the finite time survival probabilities for the classical risk model was presented (Dickson and Waters 1991). An exact analytical expression, with the help of Appell polynomials, for $\sigma(u,t)$ in the classical risk model with integer-valued claim amounts was obtained (Picard and Lefèvre 1997). Their method was numerically examined and extended (Ignatova and Kaishevb 2001; De Vylder and Goovaerts 1999; Lefèvre and Loisel 2008). Numerical evaluation methods (algorithms) for calculating the probability of ruin in finite time for the classical risk model were reviewed and compared (Dickson 1999). Explicit formulas were presented for the LT in time of the multivariate finite-time ruin probabilities (by including the deficit at ruin) of the classical risk model with phase-type claims and Sparre Andersen model with phase-type interarrivals and claims (Avram and Usábel 2003, 2004). This LT in time is with respect to the derivative of the finite-time ruin probabilities, and actually the expected probability of ruin before an exponential horizon, which can be seen as a direct approximation to the finite ruin probabilities. Recently, the explicit expressions were derived (Garcia 2005) for the finite-time survival probabilities in the classical risk model can be obtained through the inversion of the double LT of the distribution of time to ruin, when claim amounts are Erlang(2) and mixed exponential distributed.

Apart from these methods for obtaining and approximating the finite-time ruin probability, a recursive method for computing these probabilities at claim instants was proposed (Stanford and Stroïnski 1994) for the classical compound Poisson risk process with phase-type claim amounts. As explained in their paper, the ruin occurs upon the payment of claims, and hence the approach by looking at the surplus process embedded at claim instants provides an alternative and efficient approximation to the finite-time ruin probability at each instant of the claims. Explicit formulas were obtained for the probability of ruin at the arrival of the nth claim when the claims are exponentially, mixture of two exponentials and Erlang(2) distributed. The recursive algorithm was also built for the cases where claim amounts follow the mixture of m exponentials and the Erlang(n) distributions. The same problem for some non-Poisson risk models was investigated (Stanford et al. 2000).

The aim of this paper is to study more ruin related quantities at claim instants for the classical risk model. To do so, we first introduce a defective joint probability density function of three random variables, the surplus right after the $(n-1)$th claim, the surplus prior to ruin and the deficit immediately after ruin, if ruin occurs at the nth claim, and then evaluate the corresponding expected trivariate penalty function at the ruin moment. This is partially motivated by the expected discounted penalty function at ruin (Gerber and Shiu 1998), which provides an unified treatment of the time of ruin, the surplus prior to ruin and the deficit immediately after ruin. It is worth mentioning that though there are vast of papers appeared in studying this expected discounted penalty function for various risk models, all of them feature the ultimate ruin problem. In addition, to include the random variable of the surplus right after the $(n-1)$th claim (one claim before ruin) in the joint probability density function is a natural consideration following Stanford and Stroïnski (1994); in their paper, the evaluation of the ruin probability at nth claim instant depends fairly on $p_{n-1}(y;u)$, the defective probability density function corresponding to the probability of non-ruin up to $(n-1)$th claim with the remaining surplus no bigger than y and initial surplus u. Note that a recent discussion (Woo 2009) presented an explicit expression of the joint probability density of the time of ruin and these three variables, which in theory can be used to evaluation the quantities discussed in this paper. The evaluation of the finite ruin related quantities at claim instants presented here, nevertheless, provides an alternative approach.

Recursively evaluation of proposed expected penalty function at claim instants are naturally developed. One crucial component in our derivations is $p_n(y;u)$. The recursive method for computing its LT, $\hat{p}_n$, was provided (Stanford and Stroinski 1994) for some distributions within the phase-type distribution family. With the help of the Dickson-Hipp operator, we are able to obtain the explicit recursive formulas for p_n and $\hat{p}_n$ when the claim amounts follow the mixture of exponentials distribution; hence the recursion result on $\hat{p}_n$ (Stanford and Stroinski 1994) for the mixture of two exponentials is extended. The recursion procedure for evaluating the expected penalty due at the moment of the ruin is also provided.

Now we present some preliminary thoughts (Stanford and Stroinski 1994). Denote by $R_i = c\,Z_i$ the so-called inter-claim revenue earned between the $(i-1)$th and ith claims, and by $W_i = \sum_{j=1}^{i} Z_j$ the arrival time of the ith claim. It is clear that R_i's are also exponentially distributed with mean $1/\lambda$, where $\lambda = \lambda_1/c$. The surplus in (1.1) can also be written as

$$U(t) = u + \sum_{i=1}^{N(t)} (R_i - X_i)\,, \qquad t \geq 0\,,$$

where random variable $R_i - X_i$ is the difference between the premium income earned in time interval $(W_{i-1}, W_i]$ and the ith claim amount; called (Stanford and Stroinski 1994) this the "increment" between the $(i-1)$th and ith claims. Let $g(y)$ (defined on $(-\infty, \infty)$ as y can be negative) be the common density function of i.i.d. random variables $R_i - X_i$, $i = 1, 2, \ldots$, $G(y)$ be its d.f. and $\hat{g}(s) = \int_{-\infty}^{\infty} e^{-sy} g(y)dy$ be its LT.

Since the random variable $R_1 - X_1$ is the difference between the inter-claim earning and the claim amount, its LT can be easily obtained as

$$\hat{g}(s) = \frac{\lambda}{s+\lambda}\hat{f}(-s)\,. \tag{1.4}$$

Further by conditioning on one of two random variables the expression of the distribution function $G(y)$ is given by

$$G(y) = \begin{cases} \int_0^\infty [1 - e^{-\lambda(y+x)}] f(x)dx\,, & y \geq 0 \\ 1 - \int_0^\infty [F(x-y)] \lambda e^{-\lambda x} dx\,, & y < 0 \end{cases}.$$

Then the density function $g(y)$ can be obtained by differentiating $G(y)$:

$$g(y) = \begin{cases} \lambda e^{-\lambda y} \hat{f}(\lambda) = \lambda e^{-\lambda y}\, T_\lambda f(0)\,, & y \geq 0 \\ \int_0^\infty \lambda e^{-\lambda x} f(x-y)dx = \lambda\, T_\lambda f(-y)\,, & y < 0 \end{cases}, \tag{1.5}$$

where T_r is an operator (Dickson and Hipp 2001) for a real-valued function p and with respect to a complex number r, defined by

$$T_r p(y) = \int_y^\infty e^{-r(t-y)} p(t)dt\,, \qquad y \geq 0\,. \tag{1.6}$$

It is clear that $\hat{p}(s) = T_s p(0)$. Moreover, for distinct complex numbers r_1 and r_2,

$$T_{r_1} T_{r_2} p(y) = T_{r_2} T_{r_1} p(y) = \frac{T_{r_1} p(y) - T_{r_2} p(y)}{r_2 - r_1}\,, \qquad r_1 \neq r_2 \in \mathbb{C}\,,\ y \geq 0\,. \tag{1.7}$$

By these notations, (1.4) can be written as

$$\hat{g}(s) = \frac{\lambda}{s+\lambda} T_{(-s)} f(0) = \lambda\, T_{(-s)} T_\lambda f(0) + \frac{\lambda}{s+\lambda}\, T_\lambda f(0)\,. \tag{1.8}$$

Let $p_n(y; u)$ be the defective probability density function corresponding to the probability of non-ruin up to nth claim with the remaining surplus no bigger than y with the initial surplus $U(0) = u \geq 0$, i.e.,

$$p_n(y; u) = \frac{\partial}{\partial y} \Pr\{\tau \geq W_n \text{ and } U(W_n) \leq y\}\,, \qquad y \geq 0\,,$$

which is an important quantity for our further derivations. By the fact that $U(W_n) = U(W_{n-1}) + R_n - X_n$, that is, the surplus after the nth claim is the sum of the surplus after the $(n-1)$th claim and the nth increment, we have

$$p_n(y;u) = \int_0^\infty p_{n-1}(x;u)g(y-x)dx\,, \qquad n \geq 1,\; y,u \geq 0\,, \tag{1.9}$$

with $p_0(y;u)$ the delta function at initial surplus u. The LT of $p_n(y;u)$ is defined by $\hat{p}_n(s;u) = \int_0^\infty e^{-sy}p_n(y;u)dy$. Obviously, $\hat{p}_n(0;u)$ is the probability of non-ruin up to nth claim. Note that equation (1.9) leads to a recursive calculation formula of $\hat{p}_n(s;u)$ which will be presented in next section. Finally, let $\psi_n(u)$ be the probability of ruin on nth claim with initial surplus u, and it can be evaluated by

$$\psi_n(u) = \hat{p}_{n-1}(0;u) - \hat{p}_n(0;u)\,, \qquad u \geq 0\,. \tag{1.10}$$

The rest of the paper is organized as follows. In Section 2 we derive the joint density function of the surplus after the second last claim before ruin, the surplus prior to ruin and the deficit immediately after ruin for ruin occurring at the claim instants. A general recursive formula for $\hat{p}_n$ is derived and recursive formulas when claim amounts are mixture of exponentials distributed are obtained in Section 3. Then in Section 4, the expected penalty on three variates at claim instants is defined and its evaluation is discussed in the mixture of exponentials case. Finally, numerical calculations are presented in Section 5 for exponential claim amounts.

2 The joint density function of three variates

In this section, we first introduce the trivariate probability density function of the surplus after the second last claim before ruin, the surplus prior to ruin and the deficit immediately after ruin, if ruin occurs at the claim instants, then use it to derive the probability that the ruin occurs at the nth claim arrival instant, for any $n \geq 1$, and some marginal distributions.

For given initial surplus $U(0) = u \geq 0$, denote by $h_n(x,y,z;u)$ the joint probability density of the surplus right after the $(n-1)$th claim $U(W_{n-1})$, the surplus before ruin $U(W_n-)$ and the deficit immediately after ruin $|U(W_n)|$, if ruin occurs at the nth claim instant W_n. Note that $y > x$ holds for $h_n(x,y,z;u)$ as the summation of $U(W_{n-1})$ and the nth inter-claim revenue $R_n > 0$ is equal to $U(W_n-)$. Then

$$\int_0^\infty \int_0^\infty \int_0^y h_n(x,y,z;u)dxdydz = \Pr\{\tau = W_n\,|U(0) = u\} = \psi_n(u)\,, \qquad u \geq 0\,. \tag{2.1}$$

Because of the positive loading condition (1.2), $h_n(x,y,z;u)$ is a defective probability density function.

By probability formula $\Pr\{A \cap B \cap C\} = \Pr\{A\}\Pr\{B\,|\,A\}\Pr\{C\,|\,A \cap B\}$, we have, for $y > x$, the following probability

$$h_n(x,y,z;u)dx\,dy\,dz = p_{n-1}(x;u)dx \cdot \lambda\, e^{-\lambda(y-x)}dy[1-F(y)] \cdot \frac{f(y+z)dz}{1-F(y)}\,. \tag{2.2}$$

In (2.2), the first probability corresponds to the event that the ruin occurs at the nth claim (it has to be non-ruin up to $(n-1)$th claim) and the surplus $U(W_{n-1})$ is between x and $x+dx$, and the second term corresponds to the conditional probability that the surplus prior to ruin $U(W_n-)$ is between y and $y+dy$, given that the ruin occurred at the nth claim and the surplus immediately after the $(n-1)$th claim is x, while the last conditional probability ensures that the deficit at ruin, $|U(W_n)|$, is z. Hence,

$$h_n(x,y,z;u) = \begin{cases} p_{n-1}(x;u)\,\lambda\,e^{-\lambda(y-x)}\,f(y+z)\,, & y>x>0, z>0\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.3}$$

Its corresponding cumulative distribution function, denoted by $H_n(x,y,z;u)$ for $x,y,z>0$, can be written as

$$H_n(x,y,z;u) = \int_0^z \int_0^{\min\{x,y\}} \int_{x_1}^{y} p_{n-1}(x_1;u)\,\lambda\,e^{-\lambda(y_1-x_1)}\,f(y_1+z_1)dy_1dx_1dz_1\,. \tag{2.4}$$

In addition, by (2.3), we can obtain the following proposition for the probability of ruin at the nth claim instant.

Proposition 1 *The probability of ruin at the nth claim instant is given by*

$$\psi_n(u) = \lambda \int_0^\infty p_{n-1}(x;u)\,T_0T_\lambda f(x)dx\,, \qquad n\geq 1,\ u\geq 0\,. \tag{2.5}$$

In particular, $\psi_1(u) = \lambda\int_u^\infty e^{-\lambda(y-u)}[1-F(y)]dy$.

Proof By expressions (2.1) and (2.3), and some integral calculations,

$$\begin{aligned}
\psi_n(u) &= \int_0^\infty \int_0^\infty \int_x^\infty p_{n-1}(x;u)\,\lambda\,e^{-\lambda(y-x)}\,f(y+z)dydxdz \\
&= \int_0^\infty p_{n-1}(x;u) \int_x^\infty \lambda\,e^{-\lambda(y-x)}\,[1-F(y)]dydx \\
&= \int_0^\infty p_{n-1}(x;u) \int_x^\infty \left[f(y) - e^{-\lambda(y-x)}f(y)\right] dydx \\
&= \int_0^\infty p_{n-1}(x;u)\,[T_0f(x) - T_\lambda f(x)]\,dx\,,
\end{aligned}$$

which leads to (2.5) by property (1.7). When $n=1$, the result follows by the fact that $p_0(x;u)$ is the delta function at initial surplus u. $\square$

Note that the evaluation of the probability of ruin at the claim instant relies on the expression of probability density p_n, which seems difficult to be obtained. Equation (1.9) shows a recursive relationship between p_n and p_{n-1}, and a recursion formula between $\hat{p}_n(s;u)$ and $\hat{p}_{n-1}(s;u)$ is shown in the next section.

The joint density function of trivariate given by (2.2) enables us to further obtain the joint probability density functions of bivariates and univariates. Let $h_{ij,n}$ be the joint probability density function of bivariate (also a function of the initial surplus u) within three random variables $\{U(W_{n-1}), U(W_n-), |U(W_n)|\}$, $i,j=1,2,3$, if ruin occurs at the nth claim instant W_n. Then

$$h_{12,n}(x,y;u) = \begin{cases} p_{n-1}(x;u)\,\lambda\,e^{-\lambda(y-x)}\,T_0f(y)\,, & y>x>0\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{2.6}$$

$$h_{13,n}(x,z;u) = p_{n-1}(x;u)\,\lambda\,T_\lambda f(x+z)\,, \qquad x,z>0\,, \tag{2.7}$$

$$h_{23,n}(y,z;u) = \lambda\,p_{n-1} * \exp_\lambda(y;u)\,f(y+z)\,, \qquad y,z>0\,, \tag{2.8}$$

where function $\exp_\lambda$ is defined by $\exp_\lambda(x) = e^{-\lambda x}$, for $x \geq 0$, and $*$ is the notation for the convolution of two functions.

Moreover, let $h_{i,n}$ be the probability density function of the ith random variable in $\{U(W_{n-1}), U(W_n-), |U(W_n)|\}$, $i = 1, 2, 3$, if ruin occurs at the nth claim instant W_n. It is easy to obtain that

$$h_{1,n}(x; u) = p_{n-1}(x; u)\, \lambda\, T_\lambda T_0 f(x)\,, \qquad x > 0\,, \tag{2.9}$$

$$h_{2,n}(y; u) = \lambda\, p_{n-1} * \exp_\lambda(y; u)\, T_0 f(y)\,, \qquad y > 0\,, \tag{2.10}$$

$$h_{3,n}(z; u) = \lambda \int_0^\infty p_{n-1}(x; u)\, T_\lambda f(x + z) dx\,, \qquad z > 0\,. \tag{2.11}$$

We remark here that similar to (2.4) the cumulative distribution functions corresponding to densities given by (2.6)-(2.11) can be easily obtained.

3 The recursion of $\hat{p}_n$

As it will be seen that the probability of ruin occurring at the nth claim instant, $\psi_n(u)$, and other related quantities to be discussed in the following section depend immediately on the probability density function p_{n-1}, or $\hat{p}_{n-1}$. In this section we present some results regarding the recursive calculation of this important density function. First, we give the recursive formula of $\hat{p}_n(s; u)$ for claim amounts with general density function $f(x)$. We then derive the recursive formulas for $\hat{p}_n(s; u)$, $p_n(y; u)$ and $\psi_n(u)$ for the case where the claim amounts are mixture of exponentials distributed with the coefficients to be calculated recursively.

Proposition 2 *For $u \geq 0$, the recursive formula for $\hat{p}_n(s; u)$ is*

$$\hat{p}_n(s; u) = \hat{p}_{n-1}(s; u)\, \hat{g}(s) - \lambda \int_0^\infty p_{n-1}(x; u)\, T_{(-s)} T_\lambda f(x) dx\,, \qquad n \geq 1\,, \tag{3.1}$$

with $\hat{p}_0(s; u) = e^{-us}$.

Proof The proof is straightforward. Taking the LTs of both sides of recursion relations (1.9) and interchanging the order of the integration, we have

$$\begin{aligned}\hat{p}_n(s; u) &= \int_0^\infty \int_0^\infty e^{-sy} p_{n-1}(x; u)\, g(y - x) dx dy \\ &= \hat{p}_{n-1}(s; u) \hat{g}(s) - \int_0^\infty p_{n-1}(x; u) \left[\int_x^\infty e^{s(y-x)} g(-y) dy\right] dx\,,\end{aligned}$$

which is (2.8) in Stanford and Stroïnski (1994). Now by expression (1.5), for $y > 0$, $g(-y) = \lambda T_\lambda f(y)$, then the integration in square brackets above can be written as

$$\int_x^\infty e^{s(y-x)} g(-y) dy = \lambda \int_x^\infty e^{s(y-x)}\, T_\lambda f(y) dy = \lambda\, T_{(-s)} T_\lambda f(x)\,.$$

Hence, (3.1) is proved. □

Remark 1: When the claim amount follows a phase-type distribution with representation $(\boldsymbol{\alpha}, Q)$, then recursion formula (3.1) reduces to (2.6) in Stanford and Stroïnski (1994), which is

$$\hat{p}_n(s;u) = \hat{p}_{n-1}(s;u)\,\hat{g}(s) + \boldsymbol{\nu}\int_0^\infty p_{n-1}(x;u)e^{Qx}dx(s\mathbf{I}+Q)^{-1}\mathbf{q}\,, \qquad n \geq 1\,,$$

where $\boldsymbol{\nu} = \boldsymbol{\alpha}(\lambda\mathbf{I} - Q)^{-1}$ and $\mathbf{q} = -Q\mathbf{e}$. An introduction to phase-type distributions, their properties and applications in risk theory can be found (Neuts 1981).
Remark 2: In (3.1), by letting $s = 0$ we get

$$\hat{p}_n(0;u) = \hat{p}_{n-1}(0;u) - \lambda\int_0^\infty p_{n-1}(x;u)\,T_0 T_\lambda f(x)dx\,, \qquad n \geq 1\,,$$

which leads immediately to formula (2.5) in Proposition 1 according to evaluation expression (1.10).

In next subsection, we derive explicit recursions for $\hat{p}_n$, p_n and ψ_n when the claim amounts are mixture of exponentials distributed.

3.1 Recursive formulas for mixtures of exponentials

Consider the mixture of exponentials distribution with the probability density function

$$f(x) = \sum_{i=1}^N q_i\mu_i\,e^{-\mu_i x}\,, \qquad x > 0\,, \tag{3.2}$$

where the q's and μ's are positive and $\sum_{i=1}^N q_i = 1$. Without loss of generality we may assume that $\mu_1 < \mu_2 < \cdots < \mu_N$. The LT of (3.2) has the expression

$$\hat{f}(s) = \sum_{i=1}^N q_i\frac{\mu_i}{s+\mu_i}\,.$$

Due to many attractive properties, the mixture of exponentials distribution has been often used as an alternative one to the single exponential distribution in risk theory. Its simple rational fraction expression of the LT offers advantages in renewal theory as well as the stochastic processes modeling. It was showed (Kingman 1966) that any density on $(0,\infty)$ can be approximated arbitrarily closely by a function of the form (3.2).

It follows from expression (1.5) and (1.8) that

$$g(y) = \begin{cases} \lambda e^{-\lambda y}\sum_{i=1}^N q_i\frac{\mu_i}{\lambda+\mu_i}\,, & y \geq 0 \\ \lambda\sum_{i=1}^N q_i\frac{\mu_i}{\lambda+\mu_i}e^{\mu_i y}\,, & y < 0 \end{cases}\,,$$

and

$$\hat{g}(s) = \frac{\lambda}{\lambda+s}\sum_{i=1}^N q_i\frac{\mu_i}{\mu_i - s}\,. \tag{3.3}$$

In order to get the recursive formula (3.1), we need the expression of $T_{(-s)}T_\lambda f(x)$ which can be derived as

$$T_{(-s)}T_\lambda f(x) = \frac{T_{(-s)}f(x) - T_\lambda f(x)}{\lambda + s} = \sum_{i=1}^{N} \frac{q_i\mu_i}{(\mu_i - s)(\lambda + \mu_i)} e^{-\mu_i x} \,. \tag{3.4}$$

Then by (3.3) and (3.4) the recursive formula (3.1) in this case is given by

$$\hat{p}_n(s;u) = \lambda \sum_{i=1}^{N} \frac{q_i\mu_i}{\mu_i - s}\left[\frac{\hat{p}_{n-1}(s;u)}{\lambda + s} - \frac{\hat{p}_{n-1}(\mu_i;u)}{\lambda + \mu_i}\right], \qquad n \geq 1 \,, \tag{3.5}$$

with $\hat{p}_0(s;u) = e^{-us}$. We now present the recursion for $\hat{p}_n(s;u)$ in the following theorem.

Theorem 1 *For mixture of exponentials claim amounts distribution with density function given by* (3.2)*, the LT of the probability density function of non-ruin up to nth claim with remaining surplus y, $p_n(y;u)$, is given by*

$$\hat{p}_n(s;u) = \frac{\lambda^n}{(\lambda+s)^n} \sum_{i_n=1}^{N} \alpha_{i_n} \cdots \sum_{i_1=1}^{N} \alpha_{i_1} T_s T_{\mu_i}^{(n)} p_0(0) + \sum_{l=0}^{n-1} \frac{\theta_{l+1}^{(n)}}{(\lambda+s)^{n-l}} \,, \qquad n \geq 1 \,, \tag{3.6}$$

where $\alpha_i = q_i\,\mu_i$, $T_{\mu_i}^{(n)} p_0(0)$ is defined as

$$T_{\mu_i}^{(n)} p_0(0) = T_{\mu_{i_n}} T_{\mu_{i_{n-1}}} \cdots T_{\mu_{i_1}} p_0(0) \,, \qquad i_1, \cdots, i_n = 1, 2, \ldots, N \,, \tag{3.7}$$

and the n-layer coefficients $\theta_j^{(n)}$ can be calculated recursively by

$$\theta_j^{(n)} = \lambda \sum_{i_n=1}^{N} \alpha_{i_n} \left[\sum_{m=0}^{j-1} \frac{\theta_{j-m}^{(n-1)}}{(\lambda+\mu_{i_n})^{m+1}} + \frac{c_{i_n}^{(n)}}{(\lambda+\mu_{i_n})^j}\right], \qquad 1 \leq j \leq n-1 \,, \tag{3.8}$$

$$\theta_n^{(n)} = \lambda \sum_{i_n=1}^{N} \frac{\alpha_{i_n}}{\lambda+\mu_{i_n}} \hat{p}_{n-1}(\mu_{i_n};u) \,, \tag{3.9}$$

with starting value $\theta_1^{(1)} = \lambda \sum_{k=1}^{N} \frac{\alpha_k}{\lambda+\mu_k} e^{-\mu_k u}$, and lastly $c_k^{(n)}$ is defined as

$$c_{i_n}^{(n)} = \lambda^{n-1} \sum_{i_{n-1}=1}^{N} \alpha_{i_{n-1}} \cdots \sum_{i_1=1}^{N} \alpha_{i_1} T_{\mu_i}^{(n)} p_0(0) \,, \qquad 1 \leq i_n \leq N \,.$$

Proof The proof of expression (3.6) can be done by the method of induction. For $n = 1$, note that $T_s p_0(0) = \hat{p}_0(s;u)$ and then (3.5) gives

$$\begin{aligned}
\hat{p}_1(s;u) &= \lambda \sum_{i=1}^{N} \frac{q_i\mu_i}{\mu_i - s}\left[\frac{T_s p_0(0)}{\lambda+s} - \frac{T_{\mu_i} p_0(0)}{\lambda+\mu_i}\right] \\
&= \lambda \sum_{i=1}^{N} \frac{\alpha_i}{\lambda+s}\left[\frac{T_s p_0(0) - T_{\mu_i} p_0(0)}{\mu_i - s} + T_{\mu_i} p_0(0) \frac{\frac{1}{\lambda+s} - \frac{1}{\lambda+\mu_i}}{\mu_i - s}\right] \\
&= \frac{\lambda}{\lambda+s}\left[\sum_{i=1}^{N} \alpha_i T_s T_{\mu_i} p_0(0) + \sum_{i=1}^{N} \frac{\alpha_i}{\lambda+\mu_i} \hat{p}_0(\mu_i;u)\right],
\end{aligned}$$

which leads to (3.6) for $n=1$ case. Assume that (3.6) is true for $n-1$ case, that is,

$$\hat{p}_{n-1}(s;u)=\frac{\lambda^{n-1}}{(\lambda+s)^{n-1}}\sum_{i_{n-1}=1}^{N}\alpha_{i_{n-1}}\cdots\sum_{i_1=1}^{N}\alpha_{i_1}T_sT_{\mu_i}^{(n-1)}p_0(0)+\sum_{l=0}^{n-2}\frac{\theta_{l+1}^{(n-1)}}{(\lambda+s)^{n-1-l}}\,.$$

Furthermore, we can write

$$\begin{aligned}
&\frac{\frac{\hat{p}_{n-1}(s;u)}{\lambda+s}-\frac{\hat{p}_{n-1}(\mu_{i_n};u)}{\lambda+\mu_{i_n}}}{\mu_{i_n}-s}\\
&=\frac{\lambda^{n-1}}{(\lambda+s)^{n}}\sum_{i_{n-1}=1}^{N}\alpha_{i_{n-1}}\cdots\sum_{i_1=1}^{N}\alpha_{i_1}\frac{T_sT_{\mu_i}^{(n-1)}p_0(0)-T_{\mu_{i_n}}T_{\mu_i}^{(n-1)}p_0(0)}{\mu_{i_n}-s}\\
&\quad+c_{i_n}^{(n)}\left[\frac{\frac{1}{(\lambda+s)^n}-\frac{1}{(\lambda+\mu_{i_n})^n}}{\mu_{i_n}-s}\right]+\sum_{l=0}^{n-2}\theta_{l+1}^{(n-1)}\left[\frac{\frac{1}{(\lambda+s)^{n-l}}-\frac{1}{(\lambda+\mu_{i_n})^{n-l}}}{\mu_{i_n}-s}\right]\\
&=\frac{\lambda^{n-1}}{(\lambda+s)^{n}}\sum_{i_{n-1}=1}^{N}\alpha_{i_{n-1}}\cdots\sum_{i_1=1}^{N}\alpha_{i_1}T_sT_{\mu_i}^{(n)}p_0(0)+c_{i_n}^{(n)}\sum_{k=0}^{n-1}\frac{1}{(\lambda+s)^{n-k}(\lambda+\mu_{i_n})^{k+1}}\\
&\quad+\sum_{l=0}^{n-2}\theta_{l+1}^{(n-1)}\sum_{k=0}^{n-l-1}\frac{1}{(\lambda+s)^{n-l-k}(\lambda+\mu_{i_n})^{k+1}}\,. \qquad (3.10)
\end{aligned}$$

Then it follows from (3.5) and (3.10) that

$$\begin{aligned}
\hat{p}_n(s;u)&=\lambda\sum_{i_n=1}^{N}\alpha_{i_n}\left[\frac{\frac{\hat{p}_{n-1}(s;u)}{\lambda+s}-\frac{\hat{p}_{n-1}(\mu_{i_n};u)}{\lambda+\mu_{i_n}}}{\mu_{i_n}-s}+\hat{p}_{n-1}(\mu_{i_n};u)\frac{\frac{1}{\lambda+s}-\frac{1}{\lambda+\mu_{i_n}}}{\mu_{i_n}-s}\right]\\
&=\frac{\lambda^n}{(\lambda+s)^n}\sum_{i_n=1}^{N}\alpha_{i_n}\sum_{i_{n-1}=1}^{N}\alpha_{i_{n-1}}\cdots\sum_{i_1=1}^{N}\alpha_{i_1}T_sT_{\mu_i}^{(n)}p_0(0)\\
&\quad+\lambda\sum_{i_n=1}^{N}\alpha_{i_n}c_{i_n}^{(n)}\sum_{k=0}^{n-1}\frac{1}{(\lambda+s)^{n-k}(\lambda+\mu_{i_n})^{1+k}} \qquad (3.11)\\
&\quad+\lambda\sum_{i_n=1}^{N}\alpha_{i_n}\sum_{l=0}^{n-2}\theta_{l+1}^{(n-1)}\sum_{k=0}^{n-l-1}\frac{1}{(\lambda+s)^{n-l-k}(\lambda+\mu_{i_n})^{1+k}} \qquad (3.12)\\
&\quad+\frac{\lambda}{\lambda+s}\sum_{i_n=1}^{N}\frac{\alpha_{i_n}}{\lambda+\mu_{i_n}}\hat{p}_{n-1}(\mu_{i_n};u)\,. \qquad (3.13)
\end{aligned}$$

By identifying the coefficient of term $1/(\lambda+s)^j$ for each j in lines (3.11)-(3.13), and denoting it by $\theta_j^{(n)}$, for $j=1,2,\ldots,n$, we complete the proof of (3.6). □

We remark on the computational aspects of the formulas in Theorem 1 below.
Remark 1: The recursive expression (3.6) involves the computation of the composite operator T_r on function $p_0(y;u)$ which is a delta function at the initial surplus u. According to a relationship (Gerber and Shiu 2005, Eq. 10.1) between the operator T_r and the corresponding divided difference, (3.7) can be rewritten as

$$T_{\mu_i}^{(n)}p_0(0;u)=(-1)^{n-1}\hat{p}_0[\mu_{i_n},\mu_{i_{n-1}},\ldots,\mu_{i_1};u]\,, \qquad (3.14)$$

where $\hat{p}_0[\mu_{i_n}, \mu_{i_{n-1}}, \ldots, \mu_{i_1}; u]$ is the $(n-1)$th order divided difference on n points $\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_n}$, defined by

$$\hat{p}_0[\mu_{i_n}, \mu_{i_{n-1}}, \ldots, \mu_{i_1}; u] = \frac{\hat{p}_0[\mu_{i_n}, \mu_{i_{n-1}}, \ldots, \mu_{i_2}; u] - \hat{p}_0[\mu_{i_{n-1}}, \mu_{i_{n-2}}, \ldots, \mu_{i_1}; u]}{\mu_{i_n} - \mu_{i_1}} ,$$

with $\hat{p}_0[\mu_{i_j}; u] = \hat{p}(\mu_{i_j}; u) = e^{-\mu_{i_j} u}$; this relationship implies that the evaluation of (3.6) relates directly to the computation of divided differences of the exponential functions. The computationally relevant properties were investigated (McCurdy et al. 1984).
Remark 2: In the case that $\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_n}$ are distinct numbers, it follows from the well-known divided difference property (Gerber and Shiu 2005, Eq. 3.13) and (3.14) that (3.7) can be further expressed as

$$T^{(n)}_{\mu_i} p_0(0) = (-1)^n \hat{p}_0[\mu_{i_n}, \mu_{i_{n-1}}, \ldots, \mu_{i_1}] = \sum_{j=1}^{n} \frac{e^{-\mu_{i_j} u}}{\prod_{l=1, l\neq j}^{k} (\mu_{i_j} - \mu_{i_l})} .$$

Furthermore, by the definition of operator T_r on function p in (1.6), we have that

$$T_s T_r p(0) = \int_0^\infty e^{-s\,x} [T_r p(x)] dx ,$$

which shows that the Laplace inverse of $T_s T_r p(0)$ is $T_r p(x)$. In general, we have the following formula for the Laplace inverse of $[T_s(\prod_{i=1}^m T_{r_i})p](0)$:

$$\mathcal{L}^{-1} \left[T_s \left(\prod_{i=1}^{m} T_{r_i} \right) p(0) \right] = \left(\prod_{i=1}^{m} T_{r_i} \right) p(x) . \tag{3.15}$$

Thus by (3.15) and other general LT properties, we can easily derive the explicit Laplace inversion of $\hat{p}_n(s; u)$ given in (3.6) and present it in the corollary below.

Corollary 1 *For mixture of exponentials claim amounts distribution with density function given by* (3.2)*, the probability density function* $p_n(y; u)$*, for* $n \geq 1$*, is given by*

$$p_n(y; u) = \frac{\lambda^n}{(n-1)!} \sum_{i_n=1}^{N} \alpha_{i_n} \cdots \sum_{i_1=1}^{N} \alpha_{i_1} (y^{n-1} e^{-\lambda y}) * T^{(n)}_{\mu_i} p_0(y) + \sum_{l=0}^{n-1} \theta^{(n)}_{l+1} \frac{y^{n-l-1} e^{-\lambda y}}{(n-l-1)!} ,$$

where coefficients $\theta^{(n)}_j$ *are to be determined by* (3.8) *and* (3.9)*.*

Letting $s = 0$ in (3.5), we can also obtain the probability of ruin at claim instants according to (1.10), below.

Corollary 2 *For mixture of exponentials claim amounts distribution with density function given by* (3.2)*, the probability of ruin at the nth claim instant is given by*

$$\psi_n(u) = \lambda \sum_{j=1}^{N} \frac{q_j}{\lambda + \mu_j} \hat{p}_{n-1}(\mu_j; u) , \qquad n \geq 1 ,$$

where $\hat{p}_{n-1}(\mu_j; u)$ *can be obtained by* (3.6)*.*

We now consider a special case where the claim amounts are exponentially distributed with mean $1/\mu$, that is, $f(x) = \mu\, e^{-\mu x}$, for $x > 0$, and $\hat{f}(s) = \mu/(s+\mu)$. Then

$$g(y) = \begin{cases} \frac{\lambda\mu}{\lambda+\mu}\, e^{-\lambda y}\,, & y \ge 0 \\ \frac{\lambda\mu}{\lambda+\mu}\, e^{\mu y}\,, & y < 0 \end{cases}, \qquad \hat{g}(s) = \frac{\lambda\mu}{(\lambda+s)(\mu-s)}\,,$$

and the recursive formulas for $\hat{p}_n(s;u)$ and $\psi_n(u)$ are presented in the following corollary.

Corollary 3 *For exponentially distributed claim amounts with mean $1/\mu$, the LT of the probability density function of non-ruin up to nth claim with remaining surplus y, $p_n(y;u)$, is given by*

$$\hat{p}_n(s;u) = \left(\frac{\lambda\mu}{\lambda+s}\right)^n T_s\left[T_\mu^{(n)} p_0\right](0) + \sum_{l=0}^{n-1} \frac{\theta_{l+1}^{(n)}}{(\lambda+s)^{n-l}}\,, \qquad n \ge 1\,, \tag{3.16}$$

where operator $T_s[T_\mu^{(n)}] = T_s[T_\mu^n]$ and coefficients $\theta_j^{(n)}$ can be calculated recursively by

$$\begin{aligned} \theta_j^{(n)} &= \lambda\mu\left[\sum_{m=0}^{j-1} \frac{\theta_{j-m}^{(n-1)}}{(\lambda+\mu)^{m+1}} + \frac{c^{(n)}}{(\lambda+\mu)^j}\right], \qquad 1 \le j \le n-1 \\ \theta_n^{(n)} &= \frac{\lambda\mu}{\lambda+\mu}\,\hat{p}_{n-1}(\mu;u)\,, \end{aligned} \tag{3.17}$$

with starting value $\theta_1^{(1)} = e^{-\mu u}\lambda\mu/(\lambda+\mu)$, and $c^{(n)}$ is defined as

$$c^{(n)} = (\lambda\mu)^{n-1} T_\mu^{(n)} p_0(0) = \frac{(-\lambda\mu)^{n-1}}{(n-1)!}\,\frac{d^{(n-1)}}{d\mu^{(n-1)}} T_\mu p_0(0)\,, \qquad 1 \le i_n \le N\,. \tag{3.18}$$

Moreover, the probability of ruin at the nth claim instant is given by

$$\psi_n(u) = \frac{\lambda}{\lambda+\mu}\,\hat{p}_{n-1}(\mu;u)\,, \qquad n \ge 1\,. \tag{3.19}$$

Note that the last equation in (3.18) follows from a property of operator T_μ (Gerber and Shiu 2005), namely, for real-valued function p,

$$T_r^{(n)} p(x) = \frac{(-1)^{n-1}}{(n-1)!}\,\frac{d^{(n-1)}}{dr^{(n-1)}} T_r p(x)\,, \qquad r \in \mathbb{C}, x > 0\,. \tag{3.20}$$

Remark: Alternative recursions of $p_n(y;u)$ and $\psi_n(u)$ were obtained (Stanford and Stroïnski 1994) when the distribution of claim amounts is exponential; they are

$$\begin{aligned} \hat{p}_n(s;u) = e^{-\mu u} &\left[\sum_{j=1}^{n} c_j^{(n)} \left(\frac{\lambda}{\lambda+s}\right)^j (1-\Theta)^{n-j}\right. \\ &\left. + (\mu u)^n \left(\frac{\lambda}{\lambda+s}\right)^n \sum_{k=n}^{\infty} \frac{[(\mu-s)u]^{k-n}}{k!}\right], \qquad n \ge 1\,, \end{aligned} \tag{3.21}$$

and

$$\psi_n(u) = \frac{(1-\Theta)^n}{\Theta}\, e^{-\mu u} c_1^{(n)}\,, \qquad n \geq 1\,,$$

where $\Theta = \mu/(\lambda+\mu)$, and coefficients $c_j^{(n)}$ can be calculated recursively by

$$c_j^{(n)} = \Theta \left[\sum_{k=\max(1,j-1)}^{n-1} c_k^{(n-1)} + \frac{(\mu\, u)^{n-1}}{(n-1)!} \right], \qquad j = 1, \ldots, n, n \geq 2\,,$$

with $c_1^{(1)} = \Theta$. It can be verified that both recursions (3.16) and (3.21) produce the same result.

4 The expected penalty at ruin

Let $w(x, y)$, for $x, y \geq 0$, be a non-negative penalty function. Let $\delta \geq 0$ be the force of interest for valuation. Given the initial surplus u, the expected discounted penalty function at ruin (Gerber and Shiu 1998), $\phi(u)$, is defined as

$$\phi(u) = \mathbb{E}\left[e^{-\delta\, \tau}\, w(U(\tau-),\, |U(\tau)|) I(\tau < \infty) \,\Big|\, U(0) = u \right], \qquad u \geq 0\,, \tag{4.1}$$

for the surplus prior to ruin $U(\tau-)$ and the deficit at ruin $|U(\tau)|$, where $I(\cdot)$ is the indicator function. In particular, when $\delta = 0$ and $w(x, y) = 1$, (4.1) simplifies to $\psi(u)$, the ruin probability. By the natural of the finite time ruin problem discussed in this paper, instead of considering the expected discounted penalty at time of ruin $\tau > 0$, we defined the expected penalty on three ruin related quantities if ruin occurs at the nth claim instant as

$$\phi_n(u) = \mathbb{E}\left[w(U(W_{n-1}),\, U(W_n-),\, |U(W_n)|) I(\tau = W_n) \mid U(0) = u \right], \qquad u \geq 0\,, \tag{4.2}$$

where $w(x, y, z)$, for $x, y, z \geq 0$, is a non-negative penalty function on the surplus right after the $(n-1)$th claim $U(W_{n-1})$, the surplus before ruin $U(W_n-)$ and the deficit immediately after ruin $|U(W_n)|$. The joint trivariate probability density given in (2.3) enables us to evaluate the expected values defined in (4.2) as follows:

$$\begin{aligned}
\phi_n(u) &= \int_0^\infty \int_x^\infty \int_0^\infty w(x, y, z)\, p_{n-1}(x; u) \lambda\, e^{-\lambda(y-x)} f(y+z) dz dy dx \\
&= \int_0^\infty \int_x^\infty p_{n-1}(x; u)\, \lambda\, e^{-\lambda(y-x)}\, \omega(x, y) dy dx \\
&= \lambda \int_0^\infty p_{n-1}(x; u)\, \varpi(x) dx\,, \qquad u \geq 0\,,
\end{aligned} \tag{4.3}$$

where $\omega(x, y) = \int_0^\infty w(x, y, z) f(y+z) dz$ and

$$\varpi(x) = \int_x^\infty e^{-\lambda(y-x)}\, \omega(x, y) dy\,, \qquad x \geq 0 \tag{4.4}$$

which can be seen as the operator T_λ on the second variable of the bivariate function ω evaluated at x. Note that the calculation of $\phi_n(u)$ given by (4.3) depends on the expression of $p_{n-1}(x; u)$ which is discussed in previous section. In addition, if the penalty function is independent of one or two of three random variables, we can also use one of the probability density functions given by (2.6)-(2.11) to calculate the corresponding expected penalty.

Alternatively, we can evaluate $\phi_n(u)$ recursively. Indeed, if ruin occurs at the moment of the first claim arrival, by assuming that the surplus at the previous claim time is u (initial surplus) and the definition of $p_0(x; u)$, we have

$$\begin{aligned}\phi_1(u) &= \int_u^\infty \int_0^\infty w(u, y, z) \lambda\, e^{-\lambda(y-u)} f(y+z) dz dy \\ &= \int_0^\infty \lambda\, e^{-\lambda t} \int_{u+t}^\infty w(u, u+t, y-(u+t))\, f(y) dy dt \qquad u \geq 0\,, \end{aligned} \tag{4.5}$$

where the second equality is obtained by the variable change in the double integration. Note that (4.5) can have the following interpretation. The probability that the first claim occurs between time t and time $t + dt$ is $\lambda\, e^{-\lambda t} dt$ (with the operational claim intensity rate λ) and the claim amount y has to be $y > u+t$ to have ruin occured at the first claim instant (with probability $f(y)dy$). Thus by the law of iterated expectations, (4.5) is the double integration of penalty function w with respect to the claim time t and the claim amount y of the first claim. Using a similar argument to the case when ruin occurs at the nth claim instant, that is, if ruin does not occur at the first claim time t with claim amount being less than $u+t$, then the ruin must occur at the $(n-1)$th claim instant, we get the following recursive formula for $\phi_n(u)$:

$$\begin{aligned}\phi_n(u) &= \int_0^\infty \lambda\, e^{-\lambda t} \int_0^{u+t} \phi_{n-1}(u+t-y)\, f(y) dy dt \\ &= \int_u^\infty \lambda\, e^{-\lambda(x-u)} \int_0^x \phi_{n-1}(x-y)\, f(y) dy dx \\ &= \lambda\, T_\lambda [\phi_{n-1} * f](u) \\ &= \lambda \left[f * T_\lambda \phi_{n-1}(u) + T_\lambda \phi_{n-1}(0) \cdot T_\lambda f(u) \right] , \qquad u \geq 0,\, n \geq 2\,, \end{aligned}$$

with starting value $\phi_1(u)$ given by (4.5). Note that an equivalence relation (Gerber and Shiu 2005, Eq. 10.2) has been used in the last equality above.

Taking special forms of the penalty function w in equation (4.2), we are able to evaluate some finite-time ruin related quantities. It is obvious that if $w(x, y, z) = 1$, then $\phi_n(u) = \psi_n(u)$, the probability of ruin at the nth claim instant. If we assume that $w(x_1, y_1, z_1) = \mathrm{I}(x_1 \leq x, y_1 \leq y, z_1 \leq z)$, then $\phi_n(u)$ is the joint cumulative distribution function of the trivariate $H_n(x, y, z)$ given in (2.4). If we assume, for instance, that $w(x, y, z) = y^l z^m$ for $y, z > 0$, then function $\phi_n(u)$ is the joint lth moment of the surplus prior to ruin and mth moment of the deficit immediately after ruin if ruin occurs at the arrival time of the nth claim, and can be evaluated by (4.3) or directly by using the joint density function $h_{23,n}(y, z)$ given in (2.8). We illustrate some of special cases in the following section for the mixture of exponentials distributed claim amounts.

4.1 Expected penalty at ruin for mixtures of exponentials

When the claim amounts follow mixtures of exponentials with density function given by (3.2), the joint probability density function (2.3) is of the form

$$h_n(x,y,z;u) = \begin{cases} p_{n-1}(x;u)\,\lambda\, e^{\lambda x}\sum_{i=1}^N q_i\mu_i\, e^{-[(\lambda+\mu_i)y+\mu_i z]}\,, & y>x>0, z>0\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$$

We then have the following expression for $\varpi(x)$ defined in (4.4):

$$\begin{aligned}\varpi(x) &= \int_x^\infty e^{-\lambda(y-x)}\left[\int_0^\infty w(x,y,z)\sum_{i=1}^N q_i\mu_i\, e^{-\mu_i(y+z)}dz\right]dy\\ &= \sum_{i=1}^N q_i\mu_i\, e^{-\mu_i x}\int_x^\infty e^{-(\lambda+\mu_i)(y-x)}\left[\int_0^\infty w(x,y,z)e^{-\mu_i z}dz\right]dy\\ &= \sum_{i=1}^N q_i\mu_i\, e^{-\mu_i x}\int_x^\infty e^{-(\lambda+\mu_i)(y-x)}[\hat{w}(x,y;\mu_i)]dy\\ &= \sum_{i=1}^N q_i\mu_i\, e^{-\mu_i x}T_{(\lambda+\mu_i)}\hat{w}(x,x;\mu_i)\,, \qquad x\ge 0\,, \end{aligned} \tag{4.6}$$

where $\hat{w}(x,y;\mu_i)=\int_0^\infty e^{-\mu_i z}w(x,y,z)dz$ is the LT of function w with respect to the third variable valuated at μ_i, and the operator $T_{(\lambda+\mu_i)}$ in (4.6) is with respect to the second variable of $\hat{w}$. It follows that the expression for $\phi_n(u)$ given in (4.3) is:

$$\phi_n(u) = \lambda\sum_{i=1}^N q_i\mu_i\int_0^\infty e^{-\mu_i x}\,p_{n-1}(x;u)\,T_{(\lambda+\mu_i)}\hat{w}(x,x;\mu_i)dx\,, \qquad u\ge 0\,. \tag{4.7}$$

In some special cases of penalty function w, (4.7) leads to attractive formulas which can be calculated by results obtained in Section 3.1. Below are some illustrations.

(i) Penalty as a function of surplus after the second last claim before ruin. In this case, $w(x,y,z)=w(x)$ and (4.6) is simply

$$\varpi(x) = \sum_{i=1}^N \frac{q_i}{\lambda+\mu_i}\, e^{-\mu_i x}w(x)\,, \qquad x\ge 0\,,$$

and consequently (4.7) simplifies to the expected penalty on the surplus after the second last claim before ruin only, given by

$$\phi_n(u) = \lambda\sum_{i=1}^N \frac{q_i}{\lambda+\mu_i}\int_0^\infty e^{-\mu_i x}\, w(x)p_{n-1}(x;u)dx\,, \qquad u\ge 0\,. \tag{4.8}$$

Note that (4.8) can also be obtained by the density function $h_{1,n}(z;u)$ defined in (2.9).

In particular, if $w(x)=x^k$, then (4.8) is the kth moment of the surplus after the $(n-1)$th claim if ruin occurs at the nth claim instant and we denote it by $M_{1,n}^{(k)}(u)$. Since

$$\int_0^\infty e^{-\mu_i x}\, x^k p_{n-1}(x;u)dx = (-1)^k\hat{p}_{n-1}^{(k)}(\mu_i;u)\,,$$

where $\hat{p}^{(k)}_{n-1}(\mu_i;u)$ is the kth order of derivative of $\hat{p}_{n-1}(s;u)$ with respect to s valuated at μ_i, and hence we get from (4.8) that $M^{(k)}_{1,n}(u)$ is simply

$$M^{(k)}_{1,n}(u) = \lambda \sum_{i=1}^{N} \frac{q_i}{\lambda+\mu_i} (-1)^k \hat{p}^{(k)}_{n-1}(\mu_i;u)\,, \qquad u \geq 0, n, k \geq 1\,,$$

where $\hat{p}^{(k)}_{n-1}(\mu_i;u)$ can be derived from (3.6) in Theorem 1.

(ii) The joint moment of the surplus prior to ruin and the deficit right after ruin. Consider the case where $w(x,y,z) = w(y,z)$, the penalty as a function of the surplus prior to ruin and the deficit right after ruin. In this case, $\varpi(x)$ given by (4.6) becomes

$$\varpi(x) = \sum_{i=1}^{N} q_i \mu_i e^{-\mu_i x} T_{(\lambda+\mu_i)} \hat{w}(x;\mu_i)\,, \qquad x \geq 0\,, \tag{4.9}$$

and consequently formula (4.7) is of the form

$$\phi_n(u) = \lambda \sum_{i=1}^{N} q_i \mu_i \int_0^\infty e^{-\mu_i x} p_{n-1}(x;u) T_{(\lambda+\mu_i)} \hat{w}(x;\mu_i) dx\,, \qquad u \geq 0\,, n \geq 1\,. \tag{4.10}$$

Particularly, let $w(y,z) = y^l z^m$, then $\phi_n(u)$ is the joint lth moment of the surplus prior to ruin and mth moment of the deficit immediately after ruin if ruin occurs at the nth claim instant, denoted by $M^{(l,m)}_{23,n}(u)$. In this case, (4.9) reduces to

$$\begin{aligned} \varpi(x) &= \sum_{i=1}^{N} q_i \frac{m!}{\mu_i^m} e^{-\mu_i x} \int_x^\infty e^{-(\lambda+\mu_i)(y-x)} y^l dy \\ &= \sum_{i=1}^{N} q_i \frac{m!}{\mu_i^m} e^{-\mu_i x} \sum_{j=0}^{l} \frac{l!\, x^{l-j}}{(l-j)!(\lambda+\mu_i)^{j+1}}\,, \qquad x \geq 0\,, \end{aligned}$$

and (4.10) reduces to, for $n, l, m \geq 1$,

$$\begin{aligned} M^{(l,m)}_{23,n}(u) &= \lambda \sum_{i=1}^{N} q_i \frac{m!}{\mu_i^m} \sum_{j=0}^{l} \frac{l!}{(l-j)!(\lambda+\mu_i)^{j+1}} \int_0^\infty e^{-\mu_i x} x^{l-j} p_{n-1}(x;u) dx \\ &= \lambda \sum_{i=1}^{N} q_i \frac{m!}{\mu_i^m} \sum_{j=0}^{l} \frac{(-1)^{l-j} l!}{(l-j)!(\lambda+\mu_i)^{j+1}} \hat{p}^{(l-j)}_{n-1}(\mu_i;u)\,, \qquad u \geq 0\,, \end{aligned} \tag{4.11}$$

where $\hat{p}^{(l-j)}_{n-1}(\mu_i;u)$, for $j = 0, 1, \ldots, l$, with $\hat{p}^{(0)}_{n-1}(\mu_i;u) = \hat{p}_{n-1}(\mu_i;u)$, can be obtained from (3.6).

If $m = 0$, $M^{(l,m)}_{23,n}(u)$ further reduces to the lth moment of the surplus prior to ruin at the time of ruin which occurs at the nth claim instant, denoted by $M^{(l)}_{2,n}(u)$, and expression (4.11) simplifies to

$$M^{(l)}_{2,n}(u) = \lambda \sum_{i=1}^{N} q_i \sum_{j=0}^{l} \frac{(-1)^{l-j} l!}{(l-j)!(\lambda+\mu_i)^{j+1}} \hat{p}^{(l-j)}_{n-1}(\mu_i;u)\,, \qquad u \geq 0\,, l, n \geq 1\,. \tag{4.12}$$

From (4.12) we can also obtain the following recursive formula for calculating $M_{2,n}^{(l)}(u)$:

$$M_{2,n}^{(l)}(u) = \lambda \sum_{i=1}^{N} q_i \frac{(-1)^l}{\lambda+\mu_i} \hat{p}_{n-1}^{(l)}(\mu_i;u) + \frac{l}{\lambda+\mu_i} M_{2,n}^{(l-1)}(u)\,, \qquad u \geq 0\,, l \geq 2, n \geq 1\,,$$

with the expectation of the surplus immediately before ruin at the nth claim instant

$$M_{2,n}^{(1)}(u) = \lambda \sum_{i=1}^{N} q_i \left[-\frac{\hat{p}'_{n-1}(\mu_i;u)}{\lambda+\mu_i} + \frac{\hat{p}_{n-1}(\mu_i;u)}{(\lambda+\mu_i)^2} \right], \qquad u \geq 0\,, n \geq 1\,. \tag{4.13}$$

Furthermore, by setting $l = 0$, $M_{23,n}^{(l,m)}(u)$ reduces to the mth moment of the deficit at ruin if ruin occurs at the nth claim instant, denoted by $M_{3,n}^{(m)}(u)$, and expression (4.11) is simply

$$M_{3,n}^{(m)}(u) = \lambda \sum_{i=1}^{N} q_i \frac{m!}{(\lambda+\mu_i)\mu_i^m} \hat{p}_{n-1}(\mu_i;u)\,, \qquad u \geq 0, n, m \geq 1\,.$$

In particular, $M_{3,n}^{(1)}(u)$ is the expected deficit at ruin occuring at the nth claim, and it is of the form

$$M_{3,n}^{(1)}(u) = \lambda \sum_{i=1}^{N} \frac{q_i}{(\lambda+\mu_i)\mu_i} \hat{p}_{n-1}(\mu_i;u)\,, \qquad u \geq 0, n \geq 1\,. \tag{4.14}$$

Finally, by letting $l = m = 1$ in (4.11) and by expressions (4.13) and (4.14), we obtain the covariance of the surplus prior to ruin $U(W_n-)$ and the deficit immediately after ruin $|U(W_n)|$ if ruin occurs at the nth claim instant, denoted by $C_{23,n}(u)$, as follows:

$$C_{23,n}(u) = \lambda \sum_{i=1}^{N} q_i \left[-\frac{\hat{p}'_{n-1}(\mu_i;u)}{\lambda+\mu_i} + \frac{\hat{p}_{n-1}(\mu_i;u)}{(\lambda+\mu_i)^2} \right] \left(\frac{1}{\mu_i} - \lambda \sum_{j=1}^{N} \frac{q_j \hat{p}_{n-1}(\mu_j;u)}{(\lambda+\mu_j)\mu_j} \right). \tag{4.15}$$

5 Examples

In this section, we illustrate some numerical calculations when the claim amounts are exponentially distributed. First, we computer the ruin probability as a function of initial surplus u, $\psi_n(u)$, at the nth claim instant using results in Corollary 3. By comparing formulas (3.19) and (3.17), we have that

$$\psi_n(u) = \frac{\theta_n^{(n)}}{\mu}\,, \qquad u \geq 0\,.$$

By (3.17), it is easy to obtain that $\theta_1^{(1)} = e^{-\mu u}\lambda\mu/(\lambda+\mu)$, and that

$$\theta_2^{(2)} = \left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \left[u + \frac{1}{\lambda+\mu}\right] e^{-\mu u} = \left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \left[u + \frac{\kappa_{1,1}}{\lambda+\mu}\right] e^{-\mu u}\,,$$

with $\kappa_{1,1} = 1$. Then after straightforward but somewhat tedious derivations, one can get the following recursive algorithm:

$$\theta_3^{(3)} = \left(\frac{\lambda\mu}{\lambda+\mu}\right)^3 \left[\frac{u^2}{2!} + \frac{\kappa_{2,1}}{\lambda+\mu}u + \frac{\kappa_{2,2}}{(\lambda+\mu)^2}\right] e^{-\mu u}$$

with $\kappa_{2,1} = \kappa_{2,2} = 1 + \kappa_{1,1}$, and in general,

$$\theta_n^{(n)} = \left(\frac{\lambda\mu}{\lambda+\mu}\right)^n \left[\frac{u^{n-1}}{(n-1)!} + \frac{\kappa_{n-1,1}}{\lambda+\mu}\frac{u^{n-2}}{(n-2)!} + \cdots + \frac{\kappa_{n-1,n-2}}{(\lambda+\mu)^{n-2}}u + \frac{\kappa_{n-1,n-1}}{(\lambda+\mu)^{n-1}}\right] e^{-\mu u},$$

where simply

$$\kappa_{n-1,j} = 1 + \sum_{l=1}^{j} \kappa_{n-2,l}\,, \qquad j = 1, 2, \ldots, n-2\,,$$

and

$$\kappa_{n-1,n-2} = \kappa_{n-1,n-1}\,.$$

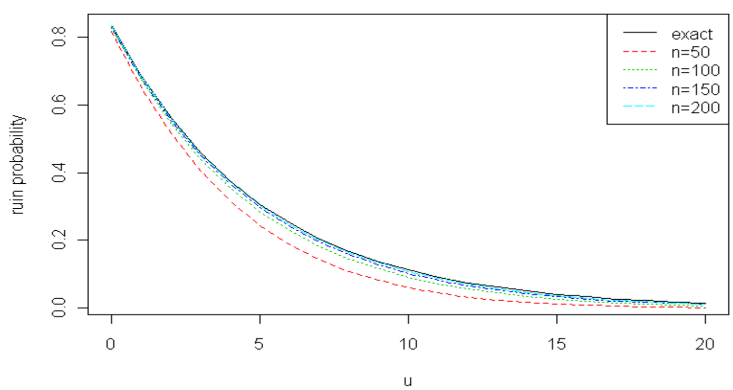

Fig. 1 Cumulative probability of ruin at claim instants.

Note that the cumulative probability of ruin, $\sum_{m=1}^{n} \psi_m(u)$, can be used to approximate the corresponding ultimate ruin probability $\psi(u)$ by the fact that

$$\psi(u) = \sum_{n=1}^{\infty} \psi_n(u)\,, \qquad u \geq 0\,.$$

Figure 1 shows the cumulative probability of ruin when $n = 50, 100, 150, 200$, respectively, and compares with the well-known ultimate ruin probability $\psi(u)$ for $0 \leq u \leq 20$ in this case, given by

$$\psi(u) = \frac{\lambda}{\mu}\, e^{-(\mu-\lambda)u}\,, \qquad u \geq 1\,.$$

The parameters in Figure 1 are $\mu = 1.2$ and $\lambda = 1$ so that the positive loading factor $(\mu/\lambda - 1)$ is 20%. As it is showed in the graph, the degree of accuracy of approximations increases as the number of claims used for cumulations increases, and is higher when u is relatively small than other values of u.

Next, we calculate, for exponentially distributed claim amounts, that the expectation of the surplus prior to ruin, $M_{2,n}^{(1)}(u)$, and the expected deficit at ruin, $M_{3,n}^{(1)}(u)$, if ruin occurs at the nth claim. In this case, expressions (4.13) and (4.14) simplify to

$$M_{2,n}^{(1)}(u) = \lambda \left[-\frac{\hat{p}'_{n-1}(\mu;u)}{\lambda+\mu} + \frac{\hat{p}_{n-1}(\mu;u)}{(\lambda+\mu)^2} \right], \qquad u \geq 0, n \geq 1, \tag{5.1}$$

$$M_{3,n}^{(1)}(u) = \frac{\lambda}{(\lambda+\mu)\mu}\, \hat{p}_{n-1}(\mu;u) = \frac{\psi_n(u)}{\mu}, \qquad u \geq 0, n \geq 1. \tag{5.2}$$

Finally, the covariance of the surplus prior to ruin and the deficit immediately after ruin if ruin occurs at the nth claim instant, given by (4.15), simplifies to, for $n \geq 1$,

$$C_{23,n}(u) = \frac{\lambda}{\mu} \left[-\frac{\hat{p}'_{n-1}(\mu;u)}{\lambda+\mu} + \frac{\hat{p}_{n-1}(\mu;u)}{(\lambda+\mu)^2} \right] \left(1 - \frac{\lambda \hat{p}_{n-1}(\mu;u)}{(\lambda+\mu)} \right), \qquad u \geq 0. \tag{5.3}$$

In order to computer these expectations and covariances, we also need to calculate $\hat{p}'_{n-1}(\mu;u)$. By (3.16), we have

$$\hat{p}_{n-1}(s;u) = \left(\frac{\lambda\mu}{\lambda+s} \right)^{n-1} T_s \left[T_\mu^{(n-1)} p_0 \right](0) + \sum_{l=0}^{n-2} \frac{\theta_{l+1}^{(n-1)}}{(\lambda+s)^{n-1-l}}. \tag{5.4}$$

Differentiating both sides of equation (5.4) gives

$$\begin{aligned} \hat{p}'_{n-1}(s;u) = {} & -(n-1)\frac{(\lambda\mu)^{n-1}}{(\lambda+s)^n}\, T_s \left[T_\mu^{(n-1)} p_0 \right](0) + \left(\frac{\lambda\mu}{\lambda+s} \right)^{n-1} \frac{d}{ds} T_s \left[T_\mu^{(n-1)} p_0 \right](0) \\ & + \sum_{l=0}^{n-2} \frac{[-(n-1-l)]\theta_{l+1}^{(n-1)}}{(\lambda+s)^{n-l}}. \end{aligned} \tag{5.5}$$

It follows from (1.6) and (3.20) that

$$\begin{aligned} T_s \left[T_\mu^{(n-1)} p_0 \right](0) &= \int_0^\infty e^{-sx} T_\mu^{(n-1)} p_0(x) dx \\ &= \int_0^\infty e^{-sx} \left[\frac{(-1)^{n-2}}{(n-2)!} \frac{d^{(n-2)}}{d\mu^{(n-2)}} T_\mu p_0(x) \right] dx \\ &= \frac{(-1)^{n-2}}{(n-2)!} \frac{d^{(n-2)}}{d\mu^{(n-2)}} \left[\int_0^\infty e^{-sx} T_\mu p_0(x) dx \right] \\ &= \frac{(-1)^{n-2}}{(n-2)!} \frac{d^{(n-2)}}{d\mu^{(n-2)}} \left[T_s T_\mu p_0(0) \right], \end{aligned} \tag{5.6}$$

and further from (1.7) that

$$T_s T_\mu p_0(0) = \frac{T_s p_0(0) - T_\mu p_0(0)}{\mu - s} = \frac{e^{-\mu s} - e^{-\mu u}}{\mu - s} = e^{-\mu u} u \sum_{k=1}^{\infty} \frac{[-u(s-\mu)]^{k-1}}{k!}. \tag{5.7}$$

Hence, putting (5.7) into (5.6), we can write the derivative of $T_s[T_\mu^{(n-1)} p_0](0)$ as

$$\frac{d}{ds} T_s \left[T_\mu^{(n-1)} p_0\right](0) = \frac{(-1)^{n-2}}{(n-2)!} \frac{d^{(n-2)}}{d\mu^{(n-2)}} \frac{d}{ds} \left[T_s T_\mu p_0(0)\right]$$
$$= \frac{(-1)^{n-2}}{(n-2)!} \frac{d^{(n-2)}}{d\mu^{(n-2)}} \left[e^{-\mu u} u \sum_{k=1}^{\infty} \frac{(-u)^{k-1}(k-1)}{k!} (s-\mu)^{k-2}\right].$$

After taking $(n-2)$th order derivative of the expression with respect to μ in square brackets above, and putting $s = \mu$, we further obtain that

$$\frac{d}{ds} T_s \left[T_\mu^{(n-1)} p_0\right](0)\Big|_{s=\mu} = \frac{u^n e^{-\mu u}}{(n-2)!} \sum_{l=0}^{n-2} \binom{n-2}{l} \frac{(-1)^{l+1}}{l+2},$$

and following from (5.5) that

$$\hat{p}'_{n-1}(\mu; u) = \left(\frac{\lambda\mu}{\lambda+\mu}\right)^{n-1} \left[\frac{-(n-1)}{\lambda+\mu} T_\mu^{(n)} p_0(0) + \frac{u^n e^{-\mu u}}{(n-2)!} \sum_{l=0}^{n-2} \binom{n-2}{l} \frac{(-1)^{l+1}}{l+2}\right]$$
$$+ \sum_{l=0}^{n-2} \frac{[-(n-1-l)]\theta_{l+1}^{(n-1)}}{(\lambda+\mu)^{n-l}}. \qquad (5.8)$$

Since $T_\mu p_0(0) = e^{-\mu u}$, then from (3.20) we have

$$T_\mu^{(n)} p_0(0) = \frac{u^{n-1}}{(n-1)!} e^{-\mu u},$$

and finally (5.8) can be rewritten as

$$\hat{p}'_{n-1}(\mu; u) = \left(\frac{\lambda\mu}{\lambda+\mu}\right)^{n-1} \frac{u^{n-1}}{(n-2)!} e^{-\mu u} \left[\frac{-1}{\lambda+\mu} + u \sum_{l=0}^{n-2} \binom{n-2}{l} \frac{(-1)^{l+1}}{l+2}\right]$$
$$+ \sum_{l=0}^{n-2} \frac{[-(n-1-l)]\theta_{l+1}^{(n-1)}}{(\lambda+\mu)^{n-l}}.$$

Then (5.1)-(5.3) can be computed accordingly.

Figure 2 shows the covariances between the surplus prior to ruin and the deficit after ruin when $n = 50, 75, 100$, respectively, for $\mu = 1.2$, $\lambda = 1$ and $0 \le u \le 20$. It is observed that three covariance curves increase first then decrease as the initial surplus increases, and the overall covariances are smaller when n is bigger or in other words the ruin occurs later.

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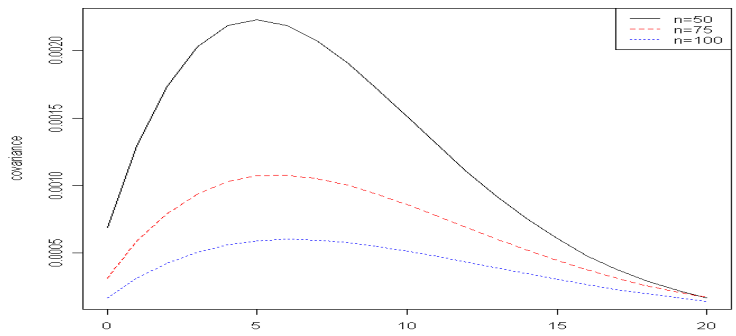

Fig. 2 Covariance between the surplus prior to ruin and the deficit after ruin.

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